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Automatic sets and Delone sets

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Abstract

Automatic sets $D \subset \mathbb{Z}^m$ are characterized by having a finite number of decimations. They are equivalently generated by fixed points of certain substitution systems, or by certain finite automata. As examples, two-dimensional versions of the Thue–Morse, Baum–Sweet, Rudin–Shapiro and paperfolding sequences are presented. We give a necessary and sufficient condition for an automatic set $D \subset \mathbb{Z}^m$ to be a Delone set in $\mathbb{R}^m$. The result is then extended to automatic sets that are defined as fixed points of certain substitutions. The morphology of automatic sets is discussed by means of examples.

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1. Introduction

A Delone (or Delaunay) set $D \subset \mathbb{R}^m$ is a set that is *uniformly discrete*, i.e. there exists an $r > 0$ such that for every $x \in \mathbb{R}^m$ the ball of radius r centred at x contains at most one element of D, and it is *relatively dense*, i.e. there exists an $R > 0$ such that for every $x \in \mathbb{R}^m$ the ball of radius R centred at x contains at least one element of D. Delone sets are the candidates for structures with long-range aperiodic order such as *quasi-crystals*, see, e.g., [1, 2].

Automatic sets $D \subset \mathbb{Z}^m$ are characterized by having a finite number of decimations (subsets of a specific type), and are equivalently generated by fixed points of certain substitution systems or by certain finite automata (hence the qualification 'automatic'), see, e.g, [3]. Apart from being considered as such, automatic sequences arise indirectly, among others, in certain cellular automata and in coarse-graining invariant structures (e.g., [4, 5]), in connection with number theory (e.g., [6]), and in the context of quasi-crystals (e.g., [7]). See [3, 8] for more extended lists of references.

In this paper, we discuss H-automatic sets, where H is an expanding endomorphism, and their relation to Delone sets. Since H-automatic sets are per definition subsets of the lattice

$\mathbb{Z}^m$, the discreteness of a subset D poses no problem at all. We state a necessary and sufficient condition for an automatic set to be a Delone set. Delone sets on lattices are also studied in [9] and [10].

The paper is organized as follows. In the next section, we present the basic facts about automatic sequences and automatic sets. In section 3, we discuss the link with substitutions and finite automata. This also provides the necessary tools for the main result which can be found in section 4. In section 5, we study Delone sets which are defined as fixed points of certain substitutions and discuss their morphology.

2. Automatic sets

We introduce the notion of an automatic set as in [3]. To this end, let $H : \mathbb{Z}^m \to \mathbb{Z}^m$ be a group-endomorphism of the additive group $(\mathbb{Z}^m, +)$, i.e. H can be considered as an $m \times m$-matrix with integer entries, and $H(x) = Hx$ (matrix vector product, where x is considered as a (column) vector in $\mathbb{Z}^m$). Moreover, we assume that H is expanding, i.e. there exists $c > 1$ such that

$$\|Hx\| \geqslant c\|x\|$$

for all $x \in \mathbb{Z}^m$ and a norm $\|\|$ that is equivalent to the usual Euclidian norm. As usual, for $R > 0$ we set

$$B_R(0) = \{x \in \mathbb{Z}^m \mid \|x\| < R\} \qquad \text{and} \qquad B_R(y) = y + B_R(0) = \{y + x \mid x \in B_R(0)\}.$$

A finite subset W of $\mathbb{Z}^m$ is called a residue set of H if for every $x \in \mathbb{Z}^m$ there exist unique $\zeta(x) \in W$ and $\kappa(x) \in \mathbb{Z}^m$ such that

$$x = \zeta(x) + H\kappa(x).$$

The maps $\zeta : \mathbb{Z}^m \to W$ and $\kappa : \mathbb{Z}^m \to \mathbb{Z}^m$ are called *residue-map* and *image-part map*, respectively. Note that the cardinality of W equals $|\det H|$.

The residue set $W = \{w_0 = 0, w_1, w_2, \ldots, w_{|\det H|-1}\}$ for H is called a *complete digit set* of H, if for every $x \in \mathbb{Z}^m$ there exists $n = n(x) \in \mathbb{N}$ such that

$$\kappa^n(x) = 0. \tag{1}$$

This is equivalent to: every $x \in \mathbb{Z}^m\backslash\{0\}$ has a finite (H, W)-*representation*, i.e. for every $x \in \mathbb{Z}^m\backslash\{0\}$ there exist unique $\omega_i \in W$, $i = 1, \ldots, n$ such that

$$x = H^{n-1}\omega_n + H^{n-2}\omega_{n-1} + \cdots + H\omega_2 + \omega_1 \tag{2}$$

and $\omega_n \neq 0$. From now on, W always denotes a complete digit set of the expanding endomorphism H. Theorem 2.2.7 in [3] guarantees, by construction, the existence of a complete digit set if the expanding constant c of H is larger than 2. However, the full determination of complete digit sets in this case, and also if $c \leqslant 2$, is an open problem. But, by lemma 2.2.3 in [3], it is sufficient to test condition (1) (or the finiteness of (2)) only for finitely many $x \in \mathbb{Z}^m$, in order to conclude that a given residue set is a complete digit set. This is how certain complete digit sets in section 5 were found. For further interesting problems and results on digit sets arising in the representation of numbers in $\mathbb{Z}$ and of complex integers, we refer to chapter 3 in [8] and chapter 7 in [11] and to the references there.

If $y \in \mathbb{Z}^m$ then $\kappa^{-1}(y)$ denotes the set $\kappa^{-1}(y) = \{Hy + w \mid w \in W\}$. Consequently, for $l \in \mathbb{N}$ one sets

$$k^{-l}(y) = \left\{H^l y + \Sigma_{i=1}^l H^{i-1}\omega_i \,\middle|\, \omega_i \in W, i = 1, \ldots, l\right\} = H^l y + \kappa^{-l}(0). \tag{3}$$

Observe that W being a residue set implies that for all $s \in \mathbb{N}$,

$$\mathbb{Z}^m = \bigcup_{y \in \mathbb{Z}^m} (H^s y + \kappa^{-s}(0)).$$

We note a trivial but important lemma.

Lemma 2.1. *Let $H : \mathbb{Z}^m \to \mathbb{Z}^m$ be expanding and let W be a complete digit set. For every $R > 0$ there exists an $N = N(R)$ such that*

$$B_R(0) \subseteq \kappa^{-N}(0).$$

Proof. For every $x \in B_R(0)$ there exists an $n = n(x)$ such that $\kappa^n(x) = 0$. If $N = \max\{n(x) \mid x \in B_R(0)\}$, then

$$\kappa^N(B_R(0)) = \{0\}.$$

Taking the inverse proves the assertion. $\square$

Let S be a finite set, which for convenience is considered as a subset of $\mathbb{Z}$. By abuse of language we call the elements of $S^{\mathbb{Z}^m}$, i.e. the maps from $\mathbb{Z}^m$ to S, *sequences*. For a sequence $f : \mathbb{Z}^m \to S$, and for $w \in W$, the (H, w)-decimation of f is defined as the sequence $\partial_w(f)$ satisfying

$$\partial_w(f)(x) = f(Hx + w). \tag{4}$$

The maps $\partial_w : S^{\mathbb{Z}^m} \to S^{\mathbb{Z}^m}, w \in W$, are called *decimations*. Repeated application of decimations to a sequence f leads to another decimation of f:

$$\partial_{\omega_n} \circ \partial_{\omega_{n-1}} \circ \cdots \circ \partial_{\omega_1}(f)(x) = f(H^n x + H^{n-1}\omega_n + H^{n-2}\omega_{n-1} + \cdots + \omega_1). \tag{5}$$

The set of all decimations of f together with f, forms the (H, W)-*kernel* of f:

$$\ker_{(H,W)}(f) = \{f\} \cup \left\{\partial_{\omega_n} \circ \partial_{\omega_{n-1}} \circ \cdots \circ \partial_{\omega_0}(f) \,\middle|\, n \in \mathbb{N}, \omega_i \in W, i = 0, \ldots, n\right\}. \tag{6}$$

We agree to simply write $\ker(f)$ if H and W are clear from the context.

Definition 2.2. *The sequence $f \in S^{\mathbb{Z}^m}$ is called (H, W)-automatic if the (H, W)-kernel is a finite set.*

Due to theorem 3.2.2 in [3], the automaticity does not depend on the choice of the residue set W. It is therefore justified to speak of an H-automatic sequence.

We conclude with the definition of an H-automatic set.

Definition 2.3. *A subset $D \subset \mathbb{Z}^m$ is called H-automatic if its characteristic sequence $\chi_D : \mathbb{Z}^m \to \{0, 1\}$, such that $\chi_D(x) = 1$ if and only if $x \in D$, is an H-automatic sequence.*

Clearly, if $f : \mathbb{Z}^m \to \{0, 1\}$ is an H-automatic sequence, then the support of f, $\mathrm{supp}(f) = \{x \mid f(x) = 1\}$ defines an H-automatic set.

3. Substitutions and finite automata

In this section, we describe how an automatic sequence is related to a fixed point of a substitution map. For an H-automatic sequence f with values in S and a complete digit set W, we define *decimation matrices* $A_w = \left(a^w_{g,h}\right) \in \{0, 1\}^{\ker(f)\times\ker(f)}$, $w \in W$ by

$$a^w_{g,h} = \begin{cases} 1 & \text{if} \quad \partial_w(g) = h \\ 0 & \text{otherwise.} \end{cases}$$

Note that every matrix A_w has precisely one 1 in each row. Since S is a subset of $\mathbb{Z}$ we can multiply the matrix A_w with the column vector $\xi = (a_h)_{h\in\ker(f)} \in S^{\ker(f)}$ from the right, then $\xi' = (a'_g)_{g\in\ker(f)} = A_w\xi$, where

$$a'_g = \sum_{h\in\ker(f)} a^w_{g,h} a_h$$

for all $g \in \ker(f)$. Using this product we can define a substitution Σ_f on the set of sequences with values in the set $S^{\ker(f)}$ in the following way. If $F : \mathbb{Z}^m \to S^{\ker(f)}$ is such a sequence, then $\Sigma_f(F) : \mathbb{Z}^m \to S^{\ker(f)}$ is defined as

$$\Sigma_f(F)(x) = A_{\zeta(x)} F(\kappa(x)) \tag{7}$$

for $x \in \mathbb{Z}^m$. Or, equivalently, $\Sigma_f(F)$ is obtained from F by putting

$$\Sigma_f(F)(Hx + w) = A_w F(x) \tag{8}$$

for all $w \in W$ and all $x \in \mathbb{Z}^m$. Then the sequence $\mathcal{F} : \mathbb{Z}^m \to S^{\ker(f)}$ defined as

$$\mathcal{F}(x) = (g(x))_{g\in\ker(f)}$$

for $x \in \mathbb{Z}^m$ is a fixed point of the substitution Σ_f, i.e. $\Sigma_f(\mathcal{F}) = \mathcal{F}$, see, e.g., [3]. As a consequence, using (8), if $x = \sum_{j=1}^n H^{j-1}\omega_j$ is the unique (H, W)-representation of $x \in \mathbb{Z}^m\backslash\{0\}$, then

$$\mathcal{F}(x) = A_{\omega_1} A_{\omega_2} \dots A_{\omega_n} \mathcal{F}(0). \tag{9}$$

Let Γ_f denote the semigroup which is generated by all products of the matrices A_w, $w \in W$. As a consequence of (9), the set of (vector) values appearing in the sequence $\mathcal{F}$ is given by the orbit of $\mathcal{F}(0)$ under the action of Γ_f, i.e. by

$$\Gamma_f\mathcal{F}(0) = \{A\mathcal{F}(0) \mid A \in \Gamma_f\}.$$

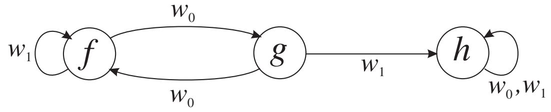
This set will play a crucial role in theorem 4.2.

The decimation matrices A_w, $w \in W$ also define a directed graph, the *kernel graph*, where the vertices correspond to the elements of $\ker(f)$, and where a vertex $g \in \ker(f)$ is connected to a vertex $h \in \ker(f)$ by a directed edge with label w, if $h = \partial_w(g)$, i.e. if $a^w_{g,h} = 1$.

For example, assume that f is an automatic $(H, \{w_0, w_1\})$-sequence, with $\ker(f) = \{f, g, h\}$ and decimation matrices

$$A_{w_0} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_{w_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

where the rows and columns of the matrices correspond to the elements f, g, h, in that order. The kernel graph associated with A_{w_0}, A_{w_1} is given as



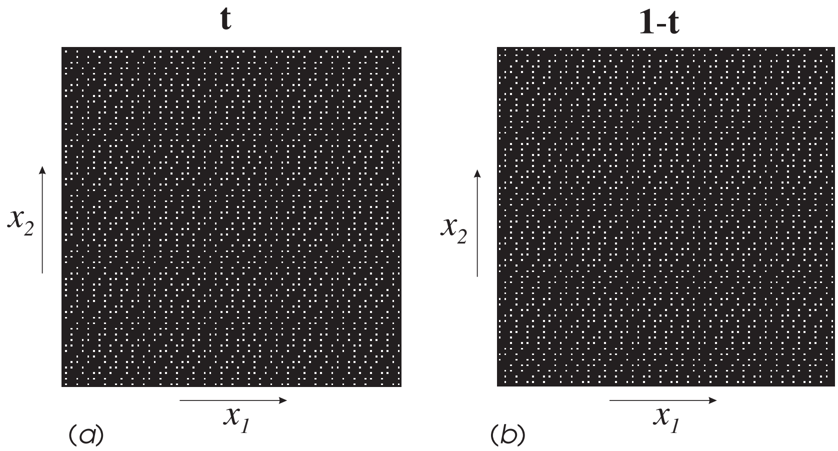


Figure 1. (*a*) The support of the two-dimensional Thue–Morse sequence $\mathbf{t}$, represented by the white points, in the square $[-30, 30] \times [-30, 30]$. (*b*) The sequence $\mathbf{1} - \mathbf{t}$ is the second element in the kernel of the sequence $\mathbf{t}$. The white points in both figures form an (H, W)-automatic set for (H, W) given in example 1.

Another interpretation of the kernel graph is that of a finite automaton that generates the sequence f, see, e.g., [3]. The idea of generating a sequence is as follows: If $x \in \mathbb{Z}^m$, $x \neq 0$, has the (H, W)-representation

$$x = \sum_{j=1}^{n} H^{j-1}\omega_j$$

then x defines a path in the directed graph. The path begins in f, follows the arrows labelled as $\omega_1, \omega_2, \ldots, \omega_n$ and terminates in an element $g \in \ker(f)$. Then the value of f at x is equal to the value of g at 0, i.e. $f(x) = g(0)$.

Example 1. The following four examples concern sequences over $\mathbb{Z}^2$ that are (H, W)-automatic counterparts of the well-known one-dimensional 2-automatic Thue–Morse, Paperfolding, Baum–Sweet and Rudin–Shapiro sequences defined over $\mathbb{N}$ (see, e.g., [6]). The expanding endomorphism on $\mathbb{Z}^2$ is defined by

$$H = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

with $W = \{w_0 = (0, 0)^T, w_1 = (1, 0)^T\}$ (T means transpose) as a complete digit set of H.

(i) *Thue–Morse sequence over* $\mathbb{Z}^2$. The one-dimensional Thue–Morse sequence $t : \mathbb{N} \to \{0, 1\}$ is defined by the recursion $t(2s) = t(s), t(2s + 1) = 1 - t(s), s \in \mathbb{N}$, starting with $t(0) = 0$. By analogy, i.e. replacing 2 by H, and 1 by w_1 one defines recursively a sequence $\mathbf{t} : \mathbb{Z}^2 \to \{0, 1\}$ by setting

$$\mathbf{t}(Hx) = \mathbf{t}(x) \qquad t(Hx + w_1) = 1 - \mathbf{t}(x)$$

and $\mathbf{t}((0, 0)^T) = 0$. Since W is a complete digit set of H, this actually defines a sequence over $\mathbb{Z}^2$. A part of this sequence is displayed in figure 1(*a*).

The kernel of $\mathbf{t}$ is given by all its decimations. We compute that $\partial_{w_0}(\mathbf{t})(x) = \mathbf{t}(Hx) = \mathbf{t}(x)$, according to the recursion.

Also, $\partial_{w_1}(\mathbf{t})(x) = \mathbf{t}(Hx + w_1) = 1 - \mathbf{t}(x)$, i.e. $\partial_{w_1}(\mathbf{t}) = \mathbf{1} - \mathbf{t}$, where $\mathbf{1}$ is the sequence with constant value 1.

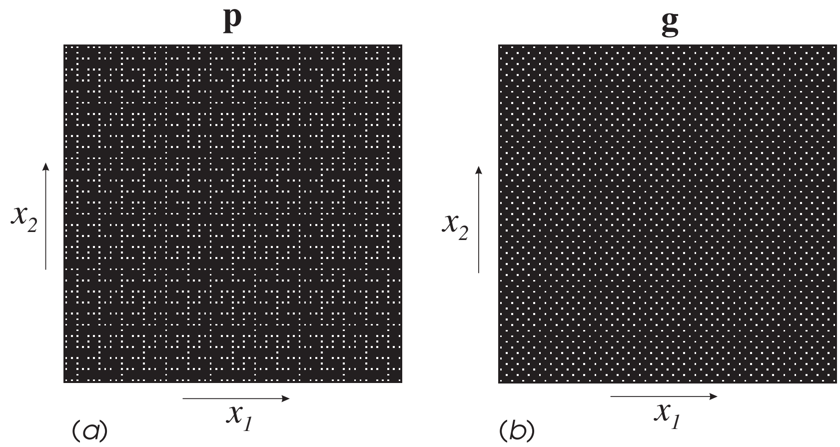


Figure 2. (*a*) The support of the two-dimensional paperfolding sequence represented by the white points in the region $[-30, 30] \times [-30, 30]$. (*b*) The sequence $\mathbf{g} = \partial_{w_0}(\mathbf{p})$ is the second nonconstant sequence in the kernel of $\mathbf{p}$.

In a similar way, it can be shown that $\partial_{w_0}\big(\partial_{w_1}(\mathbf{t})\big) = \mathbf{1} - \mathbf{t}$ and $\partial_{w_1}\big(\partial_{w_1}(\mathbf{t})\big) = \mathbf{t}$, implying that the (H, W)-kernel is given by the finite set $\{\mathbf{t}, \mathbf{1} - \mathbf{t}\}$. This shows that $\mathbf{t}$ is H-automatic. The sequence $\mathbf{1} - \mathbf{t}$ is displayed in figure 1(*b*).

The sequence $\mathcal{F}$ is thus given by $((\mathbf{t}(x), 1 - \mathbf{t}(x))^T)_{x \in \mathbb{Z}^2}$, and the decimation matrices are

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A_{w_1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

One can easily check that the corresponding substitution $\Sigma_\mathbf{t}$ has the fixed point $\mathcal{F}$. Actually, $\mathcal{F}$ can be grown by starting the substitution from $\mathcal{F}(0) = (0, 1)^T$. Also, one can easily check that the set of all possible matrix products of A_0, A_{w_1} is given by $\Gamma_\mathbf{t} = \{A_0, A_{w_1}\}$.

(ii) *Paperfolding sequence over* $\mathbb{Z}^2$. The one-dimensional paperfolding sequence p is defined by the recursion $p(4s) = 1$, $p(2s + 1) = p(s)$, $p(4s + 2) = 0$ for all $s \in \mathbb{N}$, with initial condition $p(0) = 1$. This now leads to the two-dimensional counterpart defined by

$$\mathbf{p}(H^2 x) = 1 \qquad \mathbf{p}(Hx + w_1) = \mathbf{p}(x) \qquad \mathbf{p}(H^2 x + H w_1) = 0$$

and initial condition $\mathbf{p}((0, 0)^T) = 1$. The sequence is displayed in figure 2(*a*). In order to compute the kernel of $\mathbf{p}$, observe that $\partial_{w_1}(\mathbf{p})(x) = \mathbf{p}(Hx + w_1) = \mathbf{p}(x)$, and thus $\partial_{w_1}(\mathbf{p}) = \mathbf{p}$, and that $\partial_{w_0}(\mathbf{p})(x) = \mathbf{p}(Hx)$. If $x \in H\mathbb{Z}^2 + w_1$, then $\mathbf{p}(Hx) = \mathbf{p}(H^2(H^{-1}(x - w_1)) + H w_1) = 0$, and if $x \in H\mathbb{Z}^2$, then $\mathbf{p}(Hx) = \mathbf{p}(H^2(H^{-1}x)) = 1$, due to the recursion. It follows that for the sequence $\mathbf{g} = \partial_{w_0}(\mathbf{p})$, $\partial_{w_0}(\mathbf{g}) = \mathbf{1}$ and $\partial_{w_1}(\mathbf{g}) = \mathbf{0}$, the sequence with constant value 0. The sequence $\mathbf{g}$ is the periodic sequence displayed in figure 2(*b*). As a consequence, the (H, W)-kernel of $\mathbf{p}$ is the set $\{\mathbf{p}, \mathbf{g}, \mathbf{1}, \mathbf{0}\}$ and hence $\mathbf{p}$ is H-automatic. The corresponding sequence $\mathcal{F}$ is given by $\mathcal{F} = ((\mathbf{p}(x), \mathbf{g}(x), 1, 0)^T)_{x \in \mathbb{Z}^2}$. Also here, $\mathcal{F}$ can be obtained from $\mathcal{F}(0) = (1, 1, 1, 0)^T$ by the substitution defined by the decimation matrices

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{w_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

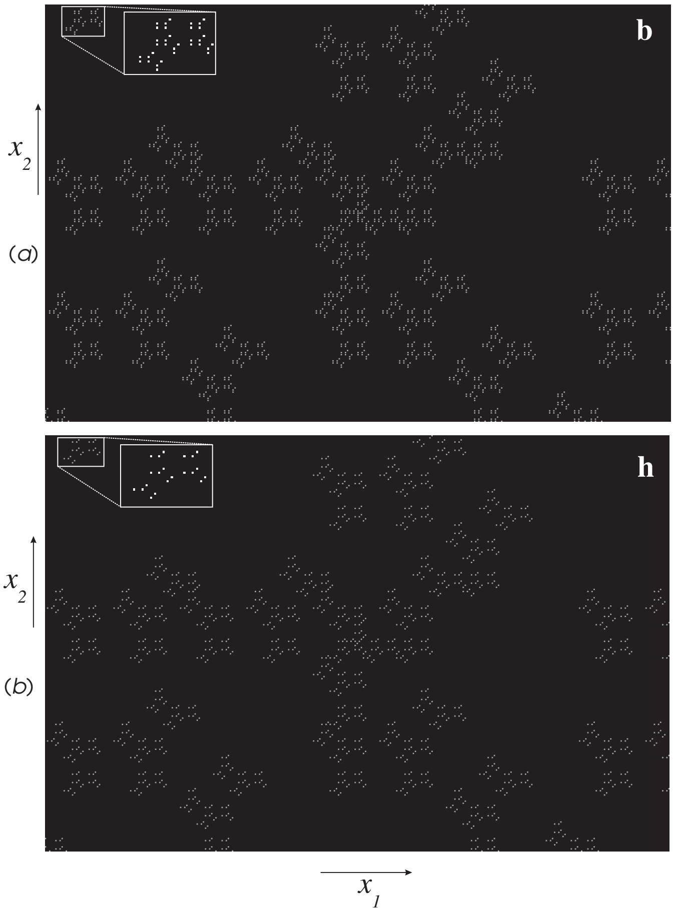


Figure 3. (*a*) Support of the two-dimensional Baum–Sweet sequence **b** in the region $[-150, 150] \times [-100, 100]$. (*b*) The sequence **h** is the other nonconstant element in the kernel **b**. The insets, in an otherwise black area, are enlargements by a factor 2 of the indicated top-left part.

Calculating the matrix products shows that

$$\Gamma_{\mathbf{p}} = \left\{A_0, A_{w_1}, A_0^2, A_{w_1}A_0, A_0A_{w_1}, A_{w_1}A_0A_{w_1}, A_{w_1}A_0^2\right\}.$$

(iii) *Baum–Sweet sequence over* $\mathbb{Z}^2$. The one-dimensional Baum–Sweet sequence b is defined recursively by $b(4s) = b(s)$, $b(2s+1) = b(s)$, $b(4s+2) = 0$ for all $s \in \mathbb{N}$, with initial condition $b(0) = 1$. This leads to the two-dimensional counterpart defined by

$$\mathbf{b}(H^2x) = \mathbf{b}(x) \qquad \mathbf{b}(Hx + w_1) = \mathbf{b}(x) \qquad \mathbf{b}(H^2x + Hw_1) = 0$$

for $x \in \mathbb{Z}^2$ and initial condition $\mathbf{b}(0) = 1$. The sequence is displayed in figure 3(*a*). Doing a similar analysis as in the previous examples, it can be shown that $\partial_{w_0}(\mathbf{b})(x) = \mathbf{b}(Hx) = 0$, if $x \in H\mathbb{Z}^2 + w_1$; and $\partial_{w_0}(\mathbf{b})(x) = \mathbf{b}(H^2(H^{-1}x)) = b(H^{-1}x)$, if $x \in H\mathbb{Z}^2$. This gives $\partial_{w_0}(\mathbf{b}) = \mathbf{h}$. Also, $\partial_{w_1}(\mathbf{b}) = \mathbf{b}$, $\partial_{w_0}(\mathbf{h}) = \mathbf{b}$ and $\partial_{w_1}(\mathbf{h}) = \mathbf{0}$. As a consequence, $\ker(\mathbf{b}) = \{\mathbf{b}, \mathbf{h}, \mathbf{0}\}$. The corresponding $\mathcal{F}$ is obtained from $\mathcal{F}(0) = (1, 1, 0)^T$ by performing the substitution induced by the decimation matrices

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad A_{w_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

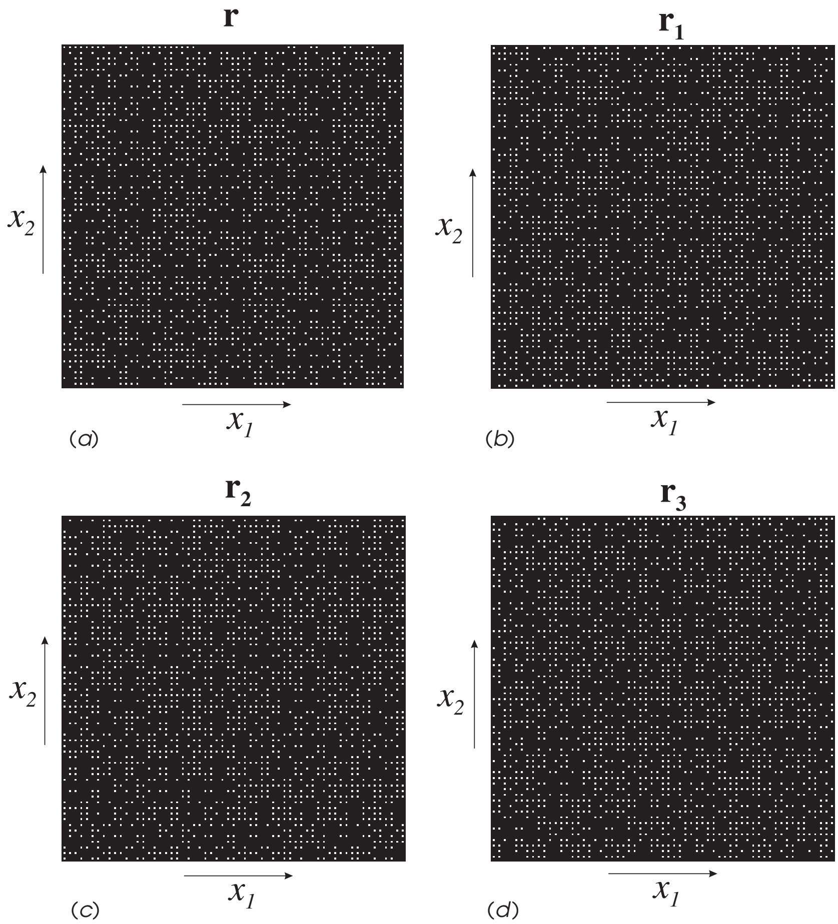


Figure 4. (*a*) Support of the two-dimensional Rudin–Shapiro sequence $\mathbf{r}$ in the domain $[-30, 30] \times [-30, 30]$. (*b*)–(*d*). The other three kernel elements of $\mathbf{r}$.

Finally,

$$\Gamma_{\mathbf{b}} = \left\{A_0, A_{w_1}, A_0^2, A_0A_{w_1}, A_{w_1}A_0, A_0A_{w_1}A_0, A_{w_1}A_0A_{w_1}\right\}.$$

(iv) *Rudin–Shapiro sequence over* $\mathbb{Z}^2$. The two-dimensional recursion defining this sequence $\mathbf{r}$ is also a straightforward adaption of its one-dimensional counterpart and is given by

$$\mathbf{r}(H^2x) = \mathbf{r}(x) \qquad \mathbf{r}(Hx + w_1) = \mathbf{r}(x) \qquad \mathbf{r}(H^2x + Hw_1 + w_1) = 1 - \mathbf{r}(Hx + w_1)$$

with $\mathbf{r}((0, 0)^T) = 0$. The sequence is displayed in figure 4(*a*). By a similar analysis as before, the (H, W)-kernel of $\mathbf{r}$ is given by the sequences $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ which are all displayed in figure 4. The corresponding decimation matrices are

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_{w_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Furthermore, $\mathcal{F}(0) = (0, 0, 1, 1)^T$, and

$$\Gamma_{\mathbf{r}} = \left\{A_0, A_{w_1}, A_{w_1}^2, A_0A_{w_1}, A_{w_1}A_0, A_{w_1}^2A_0, A_0A_{w_1}^2, A_0A_{w_1}^2A_0\right\}.$$

We conclude this section with a characterization of H-automatic sequences that will be used later. To this end we remind the reader that a sequence $(q_n)_{n\in\mathbb{N}}$ with $q_n \in S$ is called eventually periodic if there exists a positive integer d such that $q_{n+d} = q_n$ for all n sufficiently large. If $f \in S^{\mathbb{Z}^m}$ is any map, we consider the set of all decimations of level $n \in \mathbb{N}$, which is given as

$$\Delta_f(n) = \left\{\partial_{\omega_1} \circ \cdots \circ \partial_{\omega_n}(f) \,\middle|\, \omega_i \in W = \{0, w_1, \ldots, w_{|\det(H)|-1}\} \quad i = 1, \ldots, n\right\}$$

with $\Delta_f(0) = \{f\}$. The H-automatic sequences are then characterized by

Theorem 3.1 ([12]). *A sequence $f \in S^{\mathbb{Z}^m}$ is H-automatic if and only if the sequence $\Delta_f = (\Delta_f(n))_{n\in\mathbb{N}}$ is eventually periodic.*

4. Main result

A sequence $f : \mathbb{Z}^m \to \{0, 1\}$ is called a Delone sequence if it is the characteristic sequence of a Delone set, otherwise it will be called a non-Delone sequence.

Before we state the main result of this section, we state a lemma which shows how the Delone property of a sequence behaves under decimations.

Lemma 4.1.

(i) Let $f \in \{0, 1\}^{\mathbb{Z}^m}$ be a Delone sequence. If $f \in \ker_{H,W}(g)$, then g is a Delone sequence.
(ii) Let f be a non-Delone sequence. If $g \in \ker_{H,W}(f)$, then g is a non-Delone sequence.

Proof.

(i) Since $f \in \ker(g)$, there exist $n \in \mathbb{N}$, $\omega_1, \ldots, \omega_n \in W$ such that

$$f(x) = g(H^n x + H^{n-1}\omega_n + \cdots + H\omega_2 + \omega_1).$$

In other words,

$$H^n(\operatorname{supp}(f)) + H^{n-1}\omega_n + \cdots + H\omega_2 + \omega_1 \subseteq \operatorname{supp}(g).$$

Since $\operatorname{supp}(f)$ is a Delone set, it follows that $H^n(\operatorname{supp}(f)) + H^{n-1}\omega_n + \cdots + H\omega_2 + \omega_1$ is a Delone set which is contained in $\operatorname{supp}(g)$. Therefore $\operatorname{supp}(g)$ also is a Delone set. This proves the first assertion.

(ii) Follows from the first assertion by contradiction. Indeed, suppose $g \in \ker(f)$ is a Delone sequence, then (i) would imply that f is a Delone sequence, which contradicts the assumptions. □

The main result is concerned with the Delone property of H-automatic sets. If D is an H-automatic subset D, then $\chi_D : \mathbb{Z}^m \to \{0, 1\}$ is H-automatic. As indicated in the section above χ_D induces a substitution Σ_{χ_D} on the set of all sequences with values in $\{0, 1\}^{\ker(\chi_D)}$. A special element in $\{0, 1\}^{\ker(\chi_D)}$ is the zero map $\underline{0}$, defined as $\underline{0}(g) = 0$ for all $g \in \ker(\chi_D)$.

Consideration of the orbit of $\mathcal{F}_{\chi_D}(0)$ under the action of the semigroup Γ_{χ_D} of products of the decimation matrices defined by $\ker(\chi_D)$ will provide a necessary and sufficient criterion for an automatic set to be a Delone set. Namely,

Theorem 4.2. *Let $D \subset \mathbb{Z}^m$ be an H-automatic set. D is a Delone set if and only if*

$$\underline{0} \notin \Gamma_{\chi_D}(g(0))_{g\in\ker(\chi_D)}.$$

Proof. We begin with the necessity. To this end we assume that D is a Delone set, and $\underline{0} \in \Gamma_{\chi_D}(g(0))_{g\in\ker(\chi_D)}$. Since D is a Delone set, the relative denseness of D yields the existence of an $R > 0$ such that $B_R(x) \cap D \neq \emptyset$ for all $x \in \mathbb{R}^n$.

As we have seen, the sequence $\mathcal{F}_D : \mathbb{Z}^m \to \{0, 1\}^{\ker(\chi_D)}$ defined as

$$\mathcal{F}_D(x) = (g(x))_{g\in\ker(\chi_D)}$$

is a fixed point of the substitution Σ_{χ_D} induced by the H-automatic sequence χ_D. Due to the assumption $\underline{0} \in \Gamma_{\chi_D}(g(0))_{g\in\ker(\chi_D)}$, there exists an $x_0 \in \mathbb{Z}^m$ such that

$$\mathcal{F}_D(x_0) = \underline{0}.$$

Since $\mathcal{F}_D$ is a fixed point of the substitution Σ_{χ_D}, it follows that

$$\mathcal{F}_D(y) = \underline{0}$$

holds for all $y \in H^n x_0 + \kappa^{-n}(0)$ and all $n \in \mathbb{N}$. By lemma 2.1, there exists an $n_0 \in \mathbb{N}$ such that $B_R(0) \subset \kappa^{-n_0}(0)$. This implies that $B_R(H^{n_0}x_0) \subset H^{n_0}x_0 + \kappa^{-n_0}(0)$. Since $\mathcal{F}_D(y) = \underline{0}$ for all $y \in B_R(H^{n_0}x_0)$ it follows that $\chi_D(y) = 0$ for all these y. This implies that $D \cap B_R(H^{n_0}x_0) = \emptyset$, a contradiction to the Delone property of D. This shows that the Delone property implies $\underline{0} \notin \Gamma_{\chi_D}(g(0))_{g\in\ker(\chi_D)}$.

For the sufficiency we assume that $\underline{0} \notin \Gamma_{\chi_D}(g(0))_{g\in\ker(\chi_D)}$. To prove that D is a Delone set it is sufficient to show that there exists an $R > 0$ such that $B_R(x) \cap D \neq \emptyset$ for all $x \in \mathbb{Z}^m$. Due to theorem 3.1, the sequence Δ_{χ_D} is eventually periodic; let d be its minimal eventual period. There exists an $n_0 \in \mathbb{N}$ such that

$$\Delta_{\chi_D}(n_0 + n) = \Delta_{\chi_D}(n_0 + n + d)$$

for all $n \in \mathbb{N}$. Due to the fact that

$$\mathbb{Z}^m = \bigcup_{z\in\mathbb{Z}^m} \{H^s z + \kappa^{-s}(0)\}$$

for all $s \in \mathbb{N}$, there exists for every $x \in \mathbb{Z}^m$ and for every $j \in \{0, \ldots, d-1\}$ a $z_j \in \mathbb{Z}^m$ such that

$$x \in H^{n_0+j} z_j + \kappa^{-(n_0+j)}(0). \tag{10}$$

The relative denseness of D is established if we can show that for every $x \in \mathbb{Z}^m$ there exist a $j \in \{0, \ldots, d-1\}$ and a z_j such that (10) holds, and such that there is a $y \in H^{n_0+j} z_j + \kappa^{-(n_0+j)}(0)$ for which $\chi_D(y) = 1$.

For every $y \in H^{n_0+j} z_j + \kappa^{-(n_0+j)}(0)$ there exist unique $\omega_1, \ldots, \omega_{n_0+j} \in W$ such that

$$y = H^{n_0+j} z_j + H^{n_0+j-1}\omega_{n_0+j} + \cdots + H\omega_2 + \omega_1.$$

Invoking (5), this shows that $\chi_D(y)$ can be computed as

$$\chi_D(y) = \partial_{\omega_{n_0+j}} \circ \cdots \circ \partial_{\omega_1}(\chi_D)(z_j).$$

Note that $\partial_{\omega_{n_0+j}} \circ \cdots \circ \partial_{\omega_1}(\chi_D) \in \Delta_{\chi_D}(n_0 + j)$ and that for every $g \in \Delta_{\chi_D}(n_0 + j)$ there exists a $y \in H^{n_0+j} z_j + \kappa^{-(n_0+j)}(0)$ such that $\chi_D(y) = g(z_j)$. In other words,

$$\{\chi_D(y) \mid y \in H^{n_0+j} z_j + \kappa^{-(n_0+j)}(0)\} = \{g(z_j) \mid g \in \Delta_{\chi_D}(n_0 + j)\}$$

i.e. the union of the values of $g(z_j)$ for $g \in \Delta_{\chi_D}(n_0 + j)$ is the same as the union of the values of χ_D at the points $y \in H^{n_0+j} z_j + \kappa^{-(n_0+j)}(0)$.

Thus, it remains to show that

$$1 \in \bigcup_{j=0}^{d-1} \{g(z_j) \mid g \in \Delta_{\chi_D}(n_0 + j)\}.$$

Due to our assumption we have that

$$\mathcal{F}_D(y) \neq \underline{0}$$

for all $y \in \bigcup_{j=0}^{d-1}\{H^{n_0+j}z_j + \kappa^{-(n_0+j)}(0)\}$. This means that there exists an $h \in \ker(\chi_D)$ and a $y \in \bigcup_{j=0}^{d-1}\{H^{n_0+j}z_j + \kappa^{-(n_0+j)}(0)\}$ such that $h(y) = 1$.

Since y is given as $y = H^{n_0+i}z_i + H^{n_0+i-1}\omega_{n_0+i} + \cdots + H\omega_2 + \omega_1$ for an $i \in \{0, \ldots, d-1\}$ and $\omega_1, \ldots, \omega_{n_0+i} \in W$, it follows that

$$1 = h(y) = \partial_{\omega_{n_0+i}} \circ \cdots \circ \partial_{\omega_1}(h)(z_i).$$

Due to the choice of n_0, one has that $\partial_{\omega_{n_0+i}} \circ \cdots \circ \partial_{\omega_1}(h)$ belongs to $\bigcup_{j=0}^{d-1}\Delta_{\chi_D}(n_0+j)$. This shows that $1 \in \bigcup_{j=0}^{d-1}\{g(z_j) \mid g \in \Delta_{\chi_D}(n_0+j)\}$ and finishes the proof. □

Remark. If $D \subset \mathbb{R}^m$ is a Delone set, then the packing radius defined by $r_0(D) = \sup\{r > 0 \mid |B_r(x) \cap D| \leqslant 1, \forall x \in \mathbb{R}^m\}$, and the covering radius defined by $r_1(D) = \inf\{r > 0 \mid |B_r(x) \cap D| \geqslant 1, \forall x \in \mathbb{R}^m\}$ provide some characteristics of the Delone set. In particular, $2r_0$ is the minimal interpoint distance.

If D is an automatic Delone set, then being a subset of $\mathbb{Z}^m$, it follows that $r_0 \geqslant 1/2$. Furthermore, the proof of the above theorem provides an upper bound for the covering radius $r_1(D)$, namely

$$r_1(D) \leqslant \max\{\|x-y\| \mid x, y \in \kappa^{-(n_0+d-1)}\}$$

where n_0 and d are as in the proof of theorem 4.2.

Example 2. Observe from the discussion of the Thue–Morse, the Rudin–Shapiro and the paperfolding sequence that the corresponding sets $\Gamma_{\mathbf{t}}\mathcal{F}(0)$, $\Gamma_{\mathbf{r}}\mathcal{F}(0)$, $\Gamma_{\mathbf{p}}\mathcal{F}(0)$ do not contain the vector $\underline{0}$. Hence $\mathbf{t}$, $\mathbf{r}$, $\mathbf{p}$ are Delone sequences. For the Baum–Sweet sequence $\mathbf{b}$, it holds however that $\underline{0} \in \Gamma_{\mathbf{b}}\mathcal{F}(0)$, implying that $\mathbf{b}$ is not a Delone sequence as suggested by the large black regions in figure 3. Furthermore, from observation of the patterns, one obtains the following values for the packing and covering radius: $r_0(\mathrm{supp}(\mathbf{t})) = 1/2, r_1(\mathrm{supp}(\mathbf{t})) = 1$; $r_0(\mathrm{supp}(\mathbf{p})) = 1/2, r_1(\mathrm{supp}(\mathbf{p})) = 1$; $r_0(\mathrm{supp}(\mathbf{r})) = 1/2, r_1(\mathrm{supp}(\mathbf{r})) = 1$. The above-mentioned estimate for $r_1(D)$ gives $r_1(\mathrm{supp}(\mathbf{t})) \leqslant 1, r_1(\mathrm{supp}(\mathbf{p})) \leqslant \sqrt{5}, r_1(\mathrm{supp}(\mathbf{t})) \leqslant \sqrt{13}$.

The possibility that a Delone sequence may have non-Delone sequences in its kernel is already demonstrated by the paperfolding sequence which is Delone but has the identically zero sequence $\underline{0}$ in its kernel. In the next section, we present an example of a Delone sequence having a kernel element that is non-Delone and has infinite support.

5. Constructing automatic Delone sets from decimation matrices A_w

As stated in section 2, H-automatic sequences lead to a fixed point of the substitution defined in (8), based on a set of decimation matrices $A_w \in \{0,1\}^{\ker(f)}$, $w \in W$, where each A_w has precisely a single 1 in each row. From now on, we will call any matrix $A \in \{0,1\}^{N \times N}$ that contains precisely one 1 in each row, a decimation matrix.

If $\{A_w \mid w \in W\}$ is a collection of decimation matrices, then these matrices define, via equation (8), a substitution Σ on the set of sequences $F : \mathbb{Z}^m \to \{0,1\}^N$. If the substitution Σ has a fixed point $\mathcal{F} : \mathbb{Z}^m \to \{0,1\}^N$, then each of the sequences

$\mathcal{F}_j : \mathbb{Z}^m \to \{0, 1\}$, $j \in \{1, \ldots, N\}$, defined as $\mathcal{F}_j(x) = \mathcal{F}(x)_j$, i.e. the jth component of the vector sequence $\mathcal{F}$, is an H-automatic sequence, see [3] theorem 2.2.19.

The fixed points of a substitution defined by a collection of decimation matrices can be characterized in the following way:

Lemma 5.1. *Let Σ be the substitution on the sequences $F : \mathbb{Z}^m \to \{0, 1\}^N$ which is induced by decimation matrices $A_w \in \{0, 1\}^{N\times N}$, $w \in W$. $\mathcal{F}$ is a fixed point of Σ if and only if for all $x \in \mathbb{Z}^m$*

$$\mathcal{F}(x) = A_{\omega_1} \ldots A_{\omega_n} \mathcal{F}(0) \tag{11}$$

where $x = \sum_{j=1}^{n} H^{j-1}\omega_j$ is the (H, W)-representation of x. In particular,

$$\mathcal{F}(0) = A_0\mathcal{F}(0)$$

i.e. $\mathcal{F}(0)$ is a fixed point of A_0.

Proof. *Necessity*: let $\mathcal{F}$ be a fixed point of Σ. Then, according to (8), we have that

$$\mathcal{F}(0) = \Sigma(\mathcal{F})(0) = A_0\mathcal{F}(0).$$

Now let $x \in \mathbb{Z}^m\backslash\{0\}$, and $x = \sum_{j=1}^{n} H^{j-1}\omega_j$ the (H, W)-representation, then the fixed point property of $\mathcal{F}$ implies

$$\begin{aligned}\mathcal{F}(x) &= \mathcal{F}(H(H^{n-2}\omega_n + H^{n-3}\omega_{n-1} + \cdots + \omega_2) + \omega_1)\\ &= \Sigma(\mathcal{F})(H(H^{n-2}\omega_n + H^{n-3}\omega_{n-1} + \cdots + \omega_2) + \omega_1)\\ &= A_{\omega_1}\mathcal{F}(H^{n-2}\omega_n + H^{n-3}\omega_{n-1} + \cdots + \omega_2).\end{aligned}$$

A repeated application of this reasoning yields (11).

Sufficiency: now suppose that $\mathcal{F}(0)$ is a fixed point of A_0, and that

$$\mathcal{F}(x) = A_{\omega_1} \ldots A_{\omega_n}\mathcal{F}(0).$$

Then the substitution rule (8) gives $\Sigma(\mathcal{F})(0) = A_0\mathcal{F}(0) = \mathcal{F}(0)$ and, for all $x \in \mathbb{Z}^m$, $w \in W$:

$$\Sigma(\mathcal{F})(Hx + w) = A_w\mathcal{F}(x) = A_w A_{\omega_1} \ldots A_{\omega_n}\mathcal{F}(0) = \mathcal{F}(Hx + w).$$

Or, equivalently, $\Sigma\mathcal{F}(x) = \mathcal{F}(x)$ for all $x \in \mathbb{Z}^m$, meaning that $\mathcal{F}$ is a fixed point of the substitution. □

Note that $\mathcal{F}(0) = (0, \ldots, 0)^T$ and $\mathcal{F}(0) = (1, \ldots, 1)^T$ are always fixed points of A_0. They induce the constant sequences $\mathcal{F} = \mathbf{0}$ or $\mathcal{F} = \mathbf{1}$, with all 0 or 1, respectively, as fixed points of the substitution.

If $\mathcal{F}(0)$ is a fixed point of A_0 that is not of this constant type, then (11) defines the fixed point $\mathcal{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_N)^T$ of the substitution defined by the matrices A_w. All sequences $\mathcal{F}_j$ are H-automatic. It follows that theorem 4.2 can be applied to each of the $\mathcal{F}_j$-sequences to find out whether or not $\mathcal{F}_j$ is a Delone sequence, by looking at the kernel and corresponding decimation matrices proper to $\mathcal{F}_j$. This kernel consists of all $\mathcal{F}_s$ which are *reachable* from $\mathcal{F}_j$ in the A_w-defined decimation graph, i.e. for which there exists a (directed) path from $\mathcal{F}_j$ to $\mathcal{F}_s$. This is illustrated in the following:

Example 3. Consider the matrices

$$A_0 = \begin{pmatrix} 0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0 \end{pmatrix} \qquad A_{w_1} = \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0 \end{pmatrix}$$

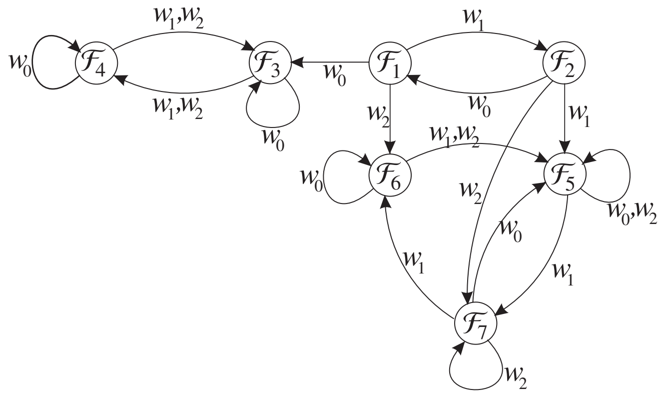


Figure 5. Decimation graph corresponding to the decimation matrices $A_{w_0} = A_0, A_{w_1}, A_{w_2}$ in example 3.

$$A_{w_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where ranking of rows and columns corresponds to $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_7$, in that order. The corresponding decimation graph is presented in figure 5. If $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_7)^T$ is a fixed point of the induced substitution, then every sequence $\mathcal{F}_j$ is (H, W)-automatic for any proper pair (H, W). Moreover, $\mathcal{F}(0)$ is given as

$$\alpha_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

with $\alpha_1, \ldots, \alpha_4 \in \{0, 1\}$.

Note that

$$\begin{aligned} \ker(\mathcal{F}_1) &= \ker(\mathcal{F}_2) = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_7\} \\ \ker(\mathcal{F}_3) &= \ker(\mathcal{F}_4) = \{\mathcal{F}_3, \mathcal{F}_4\} \\ \ker(\mathcal{F}_5) &= \ker(\mathcal{F}_6) = \ker(\mathcal{F}_7) = \{\mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7\}. \end{aligned}$$

It follows that (i) $\Gamma_{\mathcal{F}_1} = \Gamma_{\mathcal{F}_2}$ is the semigroup of all products of the transition matrices A_0, A_{w_1}, A_{w_2}, (ii) $\Gamma_{\mathcal{F}_3} = \Gamma_{\mathcal{F}_4}$ is the semigroup of all products of the 2×2-submatrices of A_0, A_{w_1}, A_{w_2} on rows and columns 3 and 4 and (iii) $\Gamma_{\mathcal{F}_5} = \Gamma_{\mathcal{F}_6} = \Gamma_{\mathcal{F}_7}$ is the set of all products of the 3×3-submatrices of $A_{w_0}, A_{w_1}, A_{w_2}$ on rows and columns 5, 6, 7.

Calculations show that taking the fixed point $\mathcal{F}$ with $\mathcal{F}(0) = (1, 1, 1, 0, 1, 0, 1)^T$ leads to $(0, \ldots, 0)^T \notin \Gamma_{\mathcal{F}_1}\mathcal{F}(0)$, and thus, according to theorem 4.2, $\mathcal{F}_1$ and $\mathcal{F}_2$ are Delone sequences.

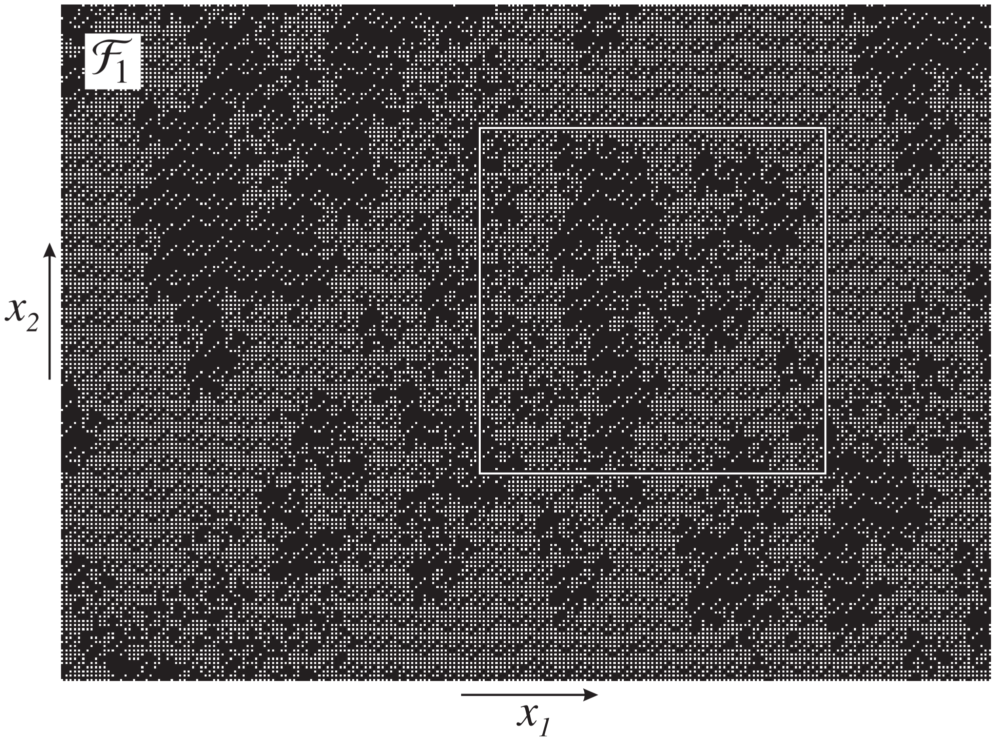


Figure 6. The automatic Delone sequence $\mathcal{F}_1$ generated from the decimation graph in figure 5 with $\mathcal{F}(0) = (1, 1, 1, 0, 1, 0, 1)^T$. Display domain $[-100, 120] \times [-80, 80]$. The white rectangle serves as a reference for figure 7.

Also, $(0, 0)^T \notin \Gamma_{\mathcal{F}_3}(\mathcal{F}_3(0), \mathcal{F}_4(0))^T$, implying that $\mathcal{F}_3$ and $\mathcal{F}_4$ also have the Delone property. However, $(0, 0, 0)^T \in \Gamma_{\mathcal{F}_5}(\mathcal{F}_5(0)\mathcal{F}_6(0)\mathcal{F}_7(0))^T$, meaning that $\mathcal{F}_5$, $\mathcal{F}_6$ and $\mathcal{F}_7$ are non-Delone.

Figure 6 shows the graphical representation of a part of the automatic Delone set corresponding to $\mathcal{F}_1$ for $H = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}$, with associated complete digit set $W = \{w_0 = (0, 0)^T, w_1 = (-1, 0)^T, w_2 = (-1, 1)^T\}$. Figure 7 shows the structure of the corresponding sequences $\mathcal{F}_2$ to $\mathcal{F}_7$. Observe that the property of the sequences being Delone or not is indeed conform to lemma 4.1. If $\mathcal{F}(0)$ is changed to $\mathcal{F}(0) = (0, 0, 0, 0, 1, 1, 1)^T$, then $\mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7$ become Delone (the constant sequence **1**) and $\mathcal{F}_3$ and $\mathcal{F}_4$ become non-Delone (the constant sequence **0**). Again conform to lemma 4.1, $\mathcal{F}_1$ and $\mathcal{F}_2$ are Delone. They are represented in figure 8. We summarize the main idea of the example in

Lemma 5.2. *Let Σ be a substitution induced by the decimation matrices $A_w \in \{0, 1\}^{N\times N}$, $w \in W$, with fixed point $\mathcal{F}$. For $\mathcal{F}_j$, $j \in \{1, \ldots, N\}$ let $\mathcal{R}_j = \{t \mid \mathcal{F}_t \in \ker(\mathcal{F}_j)\}$. Then $\mathcal{F}_j$ is a Delone sequence if and only if*

$$\underline{0} \notin \tilde{\Gamma}(\mathcal{F}_t(0))_{t\in\mathcal{R}_j}$$

where $\tilde{\Gamma}$ is the semigroup of matrix products generated by the matrices

$$\left(a^w_{uv}\right)_{u,v\in\mathcal{R}_j}$$

$w \in W$, and $(\mathcal{F}_t(0))_{t\in\mathcal{R}_j}$ is the restriction of the vector $\mathcal{F}(0)$ to $\mathcal{R}_j$.

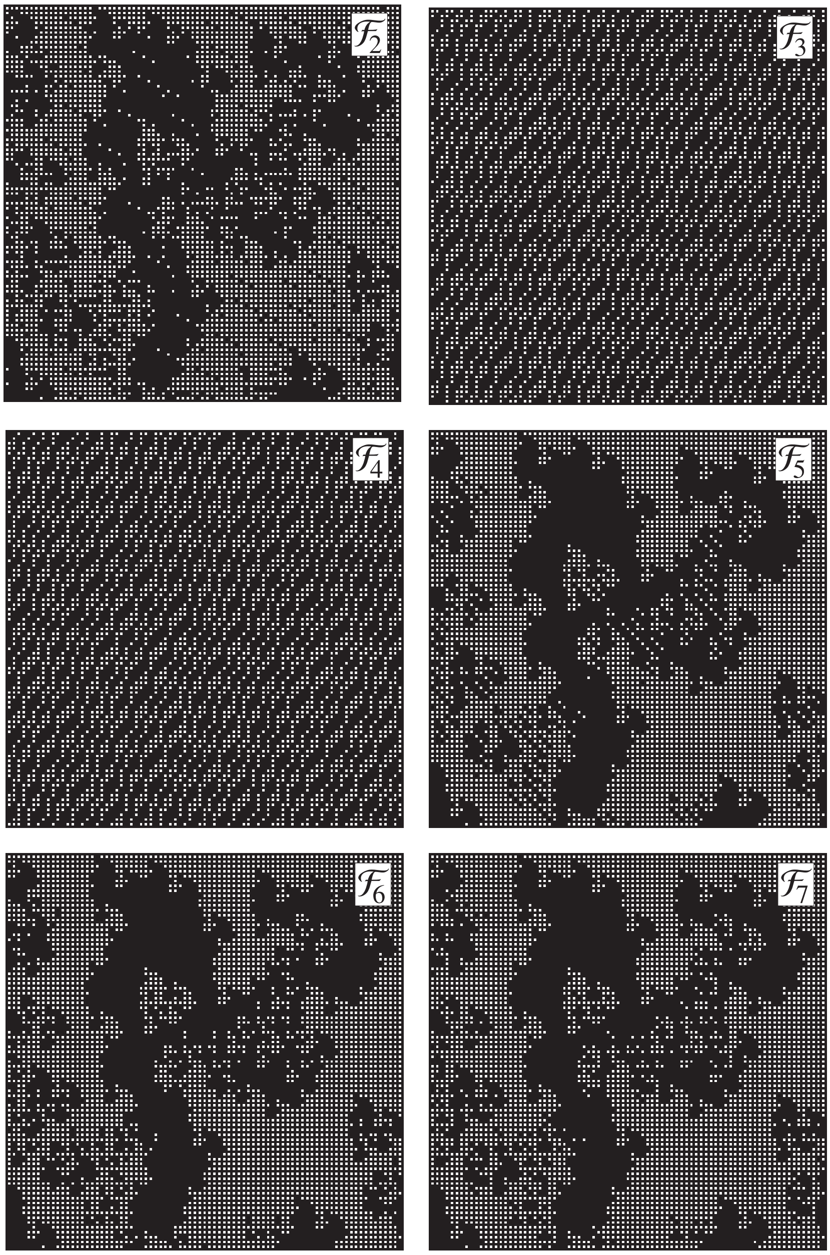


Figure 7. The automatic sequences $\mathcal{F}_j$, $j = 2, \ldots, 7$ in the white rectangle area of figure 6, to which these patterns should be compared (display domain $[0, 80] \times [-30, 50]$). $\mathcal{F}_2$, which has the Delone property, is similar to $\mathcal{F}_1$. $\mathcal{F}_3$ and $\mathcal{F}_4$, which also have Delone property, are complementary to each other, i.e. $\mathcal{F}_3 = \mathbf{1} - \mathcal{F}_4$. The sequences $\mathcal{F}_5$–$\mathcal{F}_7$, which look similar, are all non-Delone.

Example 4. Let us reconsider the substitution of example 3 (with corresponding decimation graph in figure 5), but now with the fixed point $\mathcal{F}$ defined by $\mathcal{F}(0) = (0, 0, 0, 0, 1, 0, 1)^T$ instead of $(1, 1, 1, 0, 1, 0, 1)^T$. The corresponding sequences $\mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7$ remain the same as those displayed in figure 7. But $\mathcal{F}_3$ and $\mathcal{F}_4$ now become non-Delone (the constant

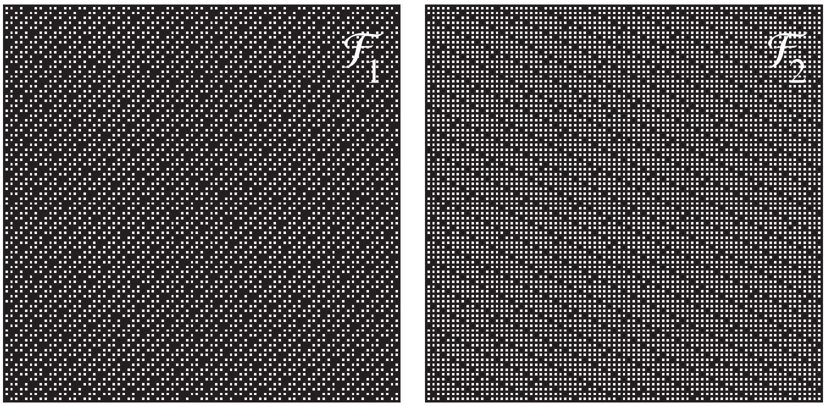


Figure 8. The automatic sequences $\mathcal{F}_1$ and $\mathcal{F}_2$ generated from the decimation graph in figure 5 with $\mathcal{F}(0) = (0, 0, 0, 0, 1, 1, 1)^T$. Display domain $[-40, 40] \times [-40, 40]$.

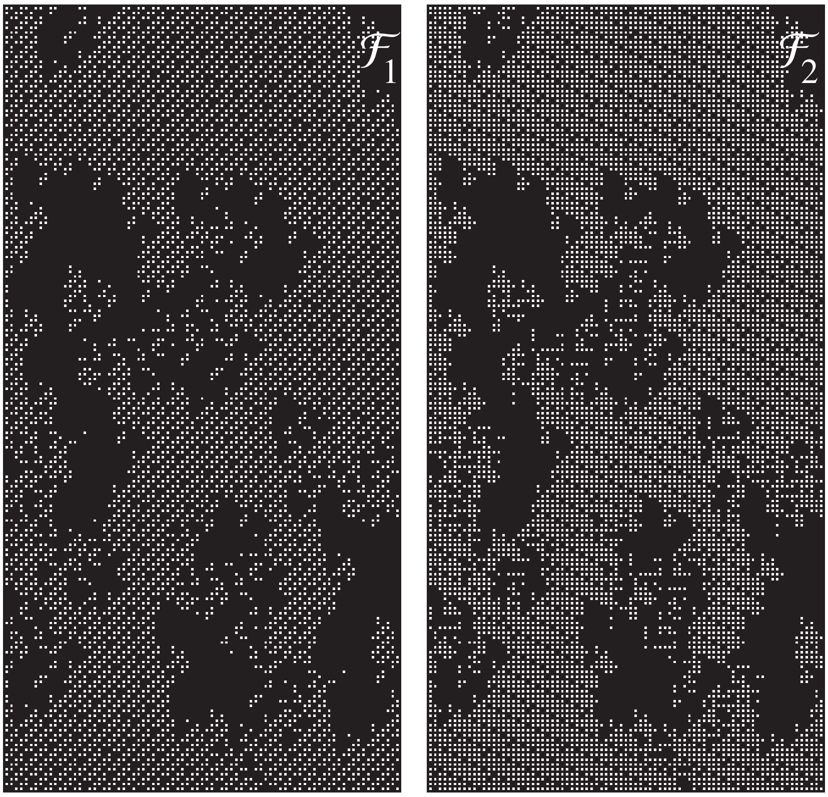


Figure 9. Graphical representation of the automatic non-Delone sequences $\mathcal{F}_1$ and $\mathcal{F}_2$ generated from the decimation graph in figure 5 with $\mathcal{F}(0) = (0, 0, 0, 0, 1, 0, 1)^T$. Display domain $[15, 95] \times [-80, 80]$. Compare the background patterns with figure 8.

0-sequence), and also $\mathcal{F}_1$ and $\mathcal{F}_2$ become non-Delone, as displayed in figure 9. Observe that the set $Q = \{\mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7\}$ is (decimation) *invariant*, i.e. every (directed) path starting in Q, remains in Q. On the other hand, the set $\{\mathcal{F}_1, \mathcal{F}_2\}$ is not invariant. In fact, there exist (directed) paths, namely w_0w_0, or w_1w_1, or w_2, such that the endpoint of the path is not in $\{\mathcal{F}_1, \mathcal{F}_2\}$, no matter whether the path starts in $\mathcal{F}_1$ or $\mathcal{F}_2$.

From this example, it seems that the fact that the sequences in the invariant set $\mathcal{Q} = \{\mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7\}$ are all non-Delone, implies that the sequences $\mathcal{F}_1$ and $\mathcal{F}_2$ from which there is a path to the invariant set $\mathcal{Q}$, are also non-Delone. That this is generally true is stated in the following theorem which also provides a converse of lemma 4.1.

Theorem 5.3. *Let $A_w \in \{0,1\}^{N\times N}$, $w \in W$, be a collection of decimation matrices and let Σ be the induced substitution. Furthermore, let $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_N)$ be a fixed point of the substitution Σ and let $Q, R \subset \{1, \ldots, N\}$ form a partition of $\{1, \ldots, N\}$ such that $\mathcal{Q} = \{\mathcal{F}_q \mid q \in Q\}$ is invariant and such that for every sequence in $\mathcal{R} = \{\mathcal{F}_r \mid r \in R\}$ there exists a (directed) path to the invariant set $\mathcal{Q}$. Then, if every sequence in the invariant set $\mathcal{Q}$ is non-Delone, all sequences in $\mathcal{R}$ are non-Delone.*

Proof. By renumbering, if necessary, we may assume that $R = \{1, \ldots, r\}$ and $Q = \{r+1, \ldots, N\}$. Then the matrices A_w are of the form

$$A_w = \begin{pmatrix} \alpha_w & \beta_w \\ \mathbf{0} & \gamma_w \end{pmatrix}$$

where α_w is an $r \times r$ matrix, γ_w is an $(N-r)\times(N-r)$ matrix and β_w is an $r \times (N-r)$ matrix and $\mathbf{0}$ denotes a proper zero matrix. Moreover, the elements of the semigroup Γ generated by all products of the matrices A_w have the same structure.

According to lemma 5.2, in order to show that all sequences in $\mathcal{R}$ are non-Delone, it is sufficient to show that there exists an $A \in \Gamma$ such that $A\mathcal{F}(0) = \underline{0}$. Since every sequence in $\mathcal{Q}$ is non-Delone, there exists an $\overline{A} \in \Gamma$ such that

$$\overline{A}\mathcal{F}(0) = \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

To ensure the existence of A we begin with the following observation. If in the decimation graph associated with the matrices A_w there is a (directed) path $\omega_1 \ldots \omega_L$ from $\mathcal{F}_u$ to $\mathcal{F}_v$, then the product $A_{\omega_1} \ldots A_{\omega_L}$ has a 1 at the (u, v)-entry. On the other hand, if the entry (u, v) of the product $A_{\omega_1} \ldots A_{\omega_L}$ is equal to 1, then the (directed) path $\omega_1 \ldots \omega_L$ which starts in $\mathcal{F}_u$ ends in $\mathcal{F}_v$.

For $\nu = (\omega_1, \ldots, \omega_L) \in W^L$ and $L \in \mathbb{N}$ one defines $T_{\nu,L} : \{1, \ldots, N\} \to \{1, \ldots, N\}$ as

$$T_{\nu,L}(u) = \text{endpoint of the (directed) path } \nu \text{ starting in } \mathcal{F}_u.$$

Using the maps $T_{\nu,L}$ one can establish the existence of an $M \in \mathbb{N}$ and $\nu \in W^M$ such that $T_{\nu,M}(R) \subseteq Q$.

Due to the properties of R and Q, for $r_1 \in R$ there exist ν_1, L_1 such that

$$T_{\nu_1,L_1}(r_1) \in Q.$$

This yields $\left|T_{\nu_1,L_1}(R) \cap R\right| < |R|$. For $r_2 \in T_{\nu_1,L_1}(R) \cap R$ there exist ν_2, L_2 such that

$$T_{\nu_2,L_2}(r_2) \in Q$$

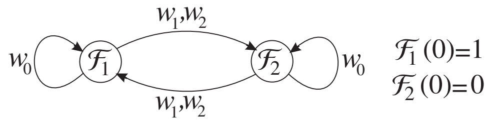


Figure 10. The automaton that generates the sequences in figures 11 and 12.

and $\left|T_{\nu_2,L_2} \circ T_{\nu_1,L_1}(R) \cap T_{\nu_1,L_1}(R) \cap R\right| < \left|T_{\nu_1,L_1}(R) \cap R\right|$. Continuing in this way finally leads to a ν and an M such that $T_{\nu,M}(R) \subseteq Q$.

Associated with the (directed) path ν is a product of the matrices A_w. This matrix is of the following form:

$$A_{\nu,M} = \begin{pmatrix} \mathbf{0} & \beta_{\nu,M} \\ \mathbf{0} & \gamma_{\nu,M} \end{pmatrix}$$

where $\mathbf{0}$ indicates proper zero matrices. Using the property of the matrix $\overline{A} \in \Gamma$ we finally arrive at

$$A_{\nu,M}\overline{A}\mathcal{F}(0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This shows that all sequences $\mathcal{F}_r \in \mathcal{R}$ are non-Delone. $\square$

Let $f : \mathbb{Z}^m \to \{0, 1\}$ be a sequence, then f^C denotes the complement of f, i.e

$$f^C(s) = 1 - f(s).$$

Lemma 5.4. *f^C is Delone if and only if $\underline{1} \notin \Gamma\mathcal{F}(0)$.*

Proof. Note that, if $\mathcal{F}(0) = (\mathcal{F}_1(0), \mathcal{F}_2(0), \ldots, \mathcal{F}_N(0))^T \in \{0, 1\}^N$ is a fixed point of A_0, then its complement $\mathcal{F}^C(0) = \left(\mathcal{F}_1^C(0), \mathcal{F}_2^C(0), \ldots, \mathcal{F}_N^C(0)\right)^T$ is also a fixed point of A_0. It means that equation (11) can be started with $\mathcal{F}^C(0)$ instead of $\mathcal{F}(0)$, generating a set of automatic sequences F_j^C which are all the complement of the corresponding $\mathcal{F}_j$. $\mathcal{F}^C$ being Delone means that $\underline{0} \notin \Gamma\mathcal{F}^C(0)$, which is equivalent to $\underline{1} \notin \Gamma\mathcal{F}(0)$. $\square$

We conclude with two remarks concerning the morphology of the Delone sets associated with Delone sequences.

Remark 1. Observe that if a sequence f is Delone, then its complement f^C may also be Delone, as illustrated by the Thue–Morse, the paperfolding and the Rudin–Shapiro sequences. It may also happen that f and f^C are complementary in their Delone-qualification: e.g., the Baum–Sweet sequence is non-Delone, but its complement is Delone. Non-Delone sequences exhibit a structure where ever growing regions of neighbouring zeros (black regions in the graphical representations) appear as one moves farther away from the origin. Sometimes, these regions (or their complementary parts in the graphical representation) form an eye-catching fractal-like pattern (repeating similar structures on larger scales). The complement f^C of a non-Delone sequence f may be Delone, also with an obvious self-similar character if this is the case for f itself (e.g., the complement of the Baum–Sweet sequence). And finally, the fact that both a sequence and its complement may be non-Delone is illustrated by the sequences $\mathcal{F}_5$–$\mathcal{F}_7$ in figure 7. Candidates for 'quasi-periodic'-like sequences which look homogenous all over space, such as the two-dimensional Thue–Morse, the paperfolding and the Rudin–Shapiro sequences, and the sequences $\mathcal{F}_3$ and $\mathcal{F}_4$ in figure 7, are to be found among those where both

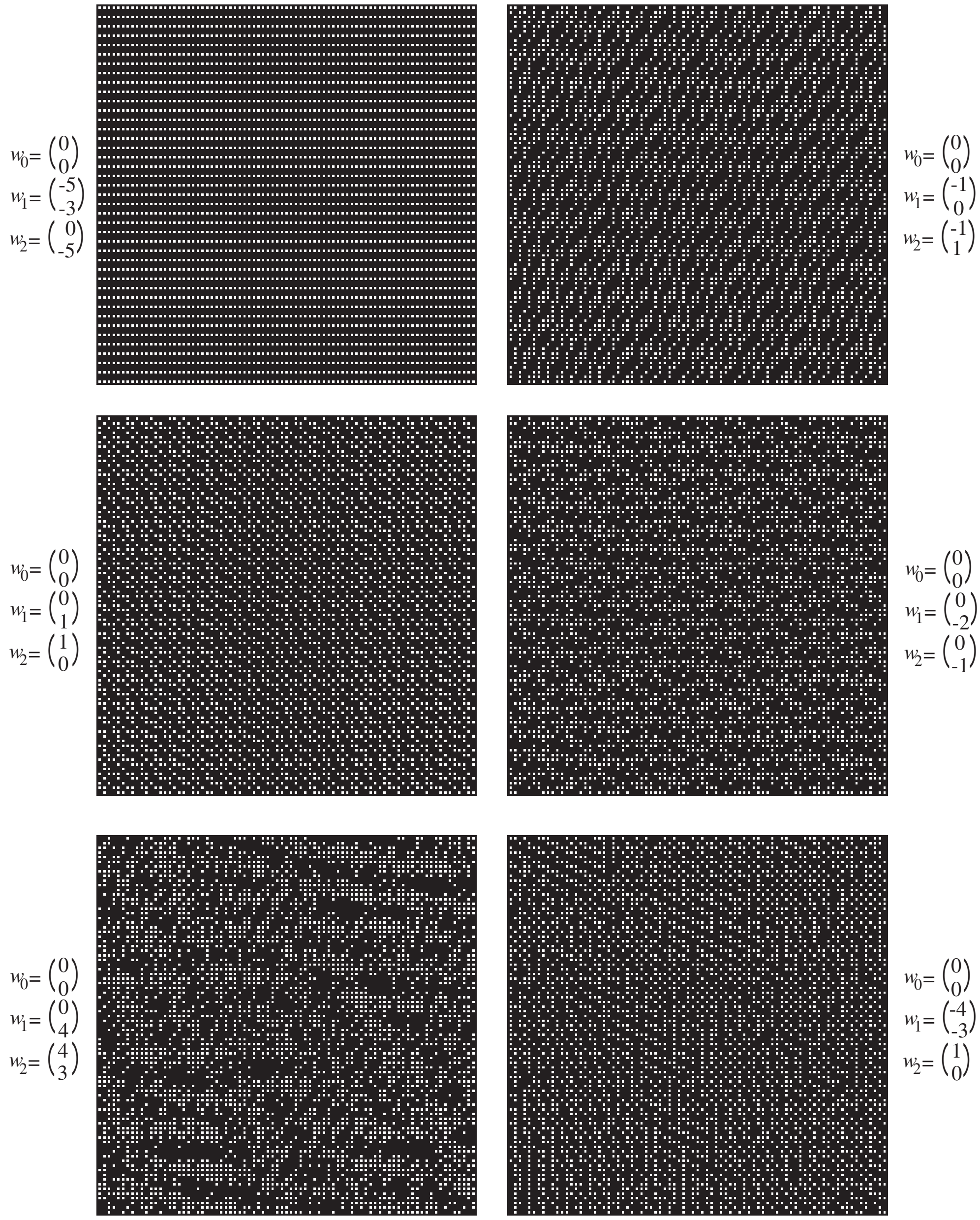


Figure 11. The automatic sequence $\mathcal{F}_1$ corresponding to the automaton displayed in figure 10, for $H = \begin{pmatrix}-1 & -1\\ 2 & -1\end{pmatrix}$ and different complete digit sets. All sequences are Delone, and so are their complements which correspond to the sequence $\mathcal{F}_2$. Display domain $[-40, 40] \times [-40, 40]$.

the sequence and its complement are Delone. This condition is however not sufficient for an homogenous pattern, as the sequence in figure 6 illustrates, and where the corresponding kernel contains non-Delone sequences. We *conjecture* that an homogenous aperiodic pattern for the graphical representation of a sequence occurs if the sequence and its complement are

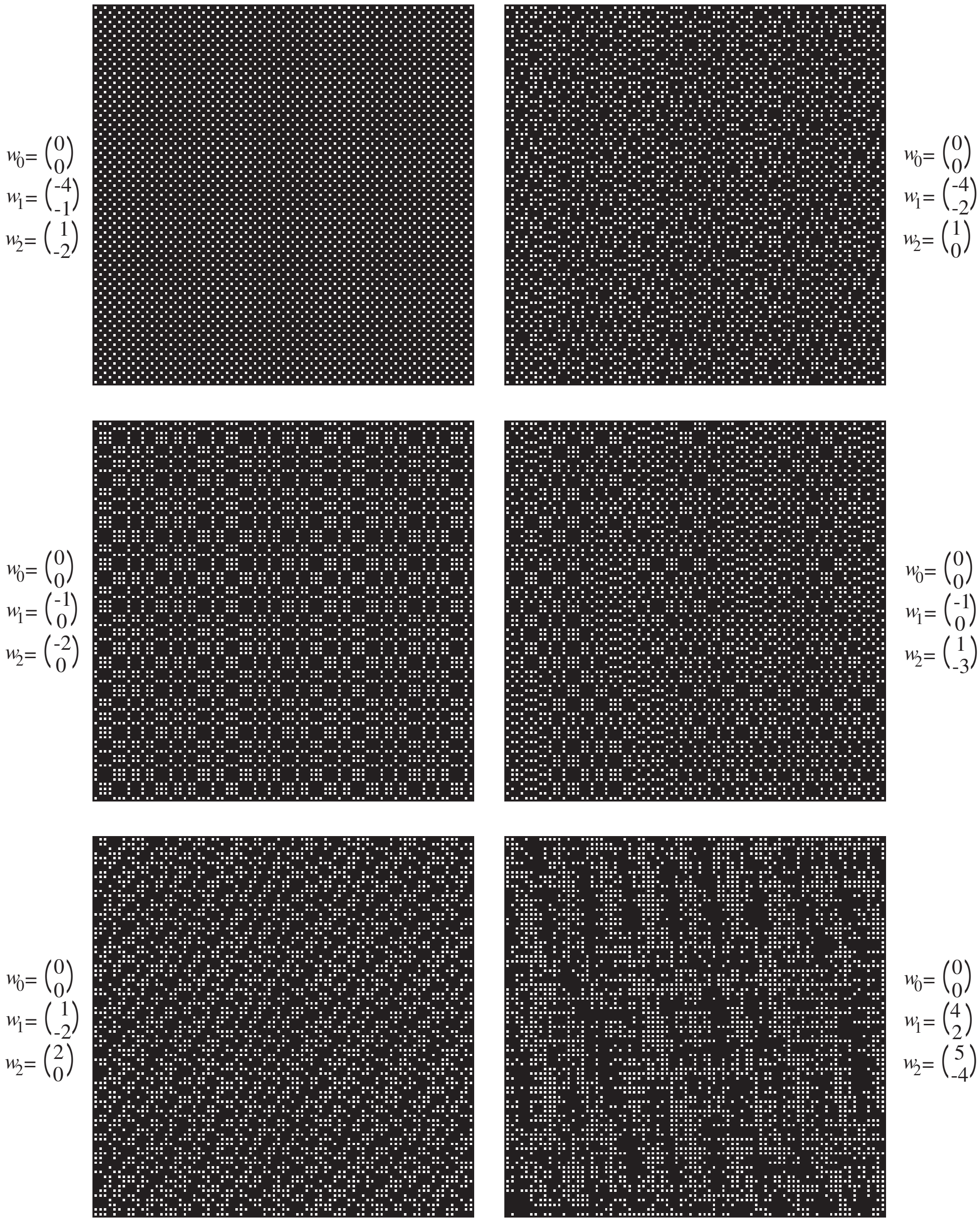


Figure 12. The automatic sequence $\mathcal{F}_1$ corresponding to the automaton displayed in figure 10, for $H = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix}$ and different complete digit sets. All sequences are Delone, and so are their complements which correspond to the sequence $\mathcal{F}_2$. Display domain $[-40, 40] \times [-40, 40]$.

Delone, *and* when the associated kernel-graph is strongly connected (i.e. every sequence in the kernel is some decimation of all the other sequences in the kernel).

Remark 2. Observe from the previous developments that the fact that a sequence appearing in a kernel-graph is Delone or not only depends on the set of decimation matrices $\{A_w \mid w \in W\}$

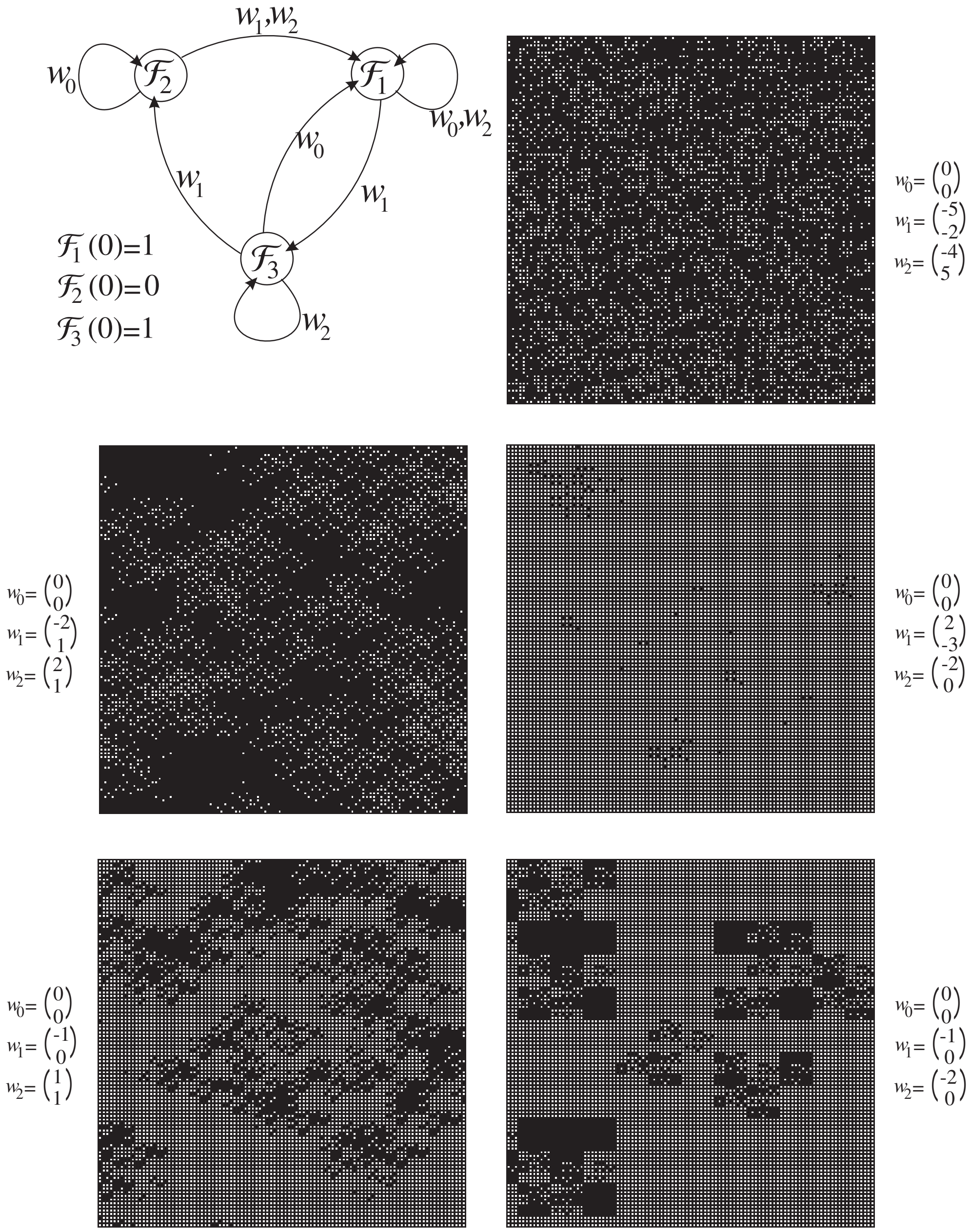


Figure 13. The automatic sequence $\mathcal{F}_1$ corresponding to the automaton displayed on top, for $H = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix}$ and different complete digit sets. All sequences are non-Delone, and so are their complements. The corresponding sequences $\mathcal{F}_2$ and $\mathcal{F}_3$ are similar in nature. Display domain $[-50, 50] \times [-50, 50]$.

and the fixed point of A_0 that is taken as $\mathcal{F}(0)$, and not on the specific expanding endomorphism H and the associated complete digit set W. The actual sequences, and by consequence also their graphical representations, do however depend on the (H, W)-specification. Figure 11 displays the automatic sequence $\mathcal{F}_1$, generated by the automaton shown in figure 10, for different complete digit sets W for the same $H = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}$. Figure 12 does the same for the matrix $H = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix}$. All these examples are Delone, and so are the complementary sequences. Figure 13 shows sequences for the automaton displayed on top and for $H = \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix}$. These sequences are all non-Delone, and so are their complements. But observe that this is not always visually clear from the limited displayed part of the corresponding graphical representation: one has to look in an area much farther away from the origin in order to find larger black areas.

Acknowledgments

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